Math 3235 Probability Theory 3/14/23

Chapter 4 : Probability Generating Functions.

Generating function:

a_n is a sequence of real numbers

$A(s) = a_0 + a_1 s + a_2 s^2 + \dots + a_n s^n + \dots$

if there exists a constant c

such that

$|a_n| \leq c^n = e^{(\ln c)n}$

Then

$A(s)$ is well defined for $|s| < \frac{1}{c}$

well defined

$\sum_n |a_n| |s|^n < +\infty$

Under our condition

$$|a_n||s|^n < c^n\left(\frac{1}{c}\right)^n = 1$$

$A(s)$ defined for $|s| < \frac{1}{c}$

$$\frac{d}{ds}A(s) = \sum_{n=0}^{\infty} n a_n s^{n-1}$$

$$\frac{d^k A}{ds^k}(s) = \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n s^{n-k}$$

Examples

$a_n = \frac{1}{n!}$ $\qquad A(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} = e^{-s}$

$a_n = \binom{N}{n}$ $\quad n \le N \qquad 0$ otherwise

$$A(s) = \sum_{n=0}^{N} a_n s^n = (1+s)^N$$

Generating function:

$$a_n \longrightarrow A(s)$$

$$A(s) = \sum_{n=0}^{\infty} a_n s^n$$

$$A(0) = a_0$$

$$A'(0) = a_1$$

$$A''(s) = \sum_{\substack{n=2 \\ n=0}}^{\infty} n(n-1) s^{n-2} a_n$$

$$A^{(n)}(0) = n!\, a_n$$

$$a_n = \frac{A^{(n)}(0)}{n!}$$

$A(n)$ generating f. $\longrightarrow$ a_n sequence.

Example

$$a_n = (-1)^n$$

$$A(s) = \frac{1}{(1+s)}$$

$A(s)$ is defined for $|s| < 1$

if $|s| > 1$

$$\sum_n |a_n|\, |s|^n = +\infty$$

$$\sum_n |a_n| = +\infty \implies \sum_n a_n \nearrow^{+\infty} \searrow \text{does not exists}$$

if $\sum_n a_n < +\infty$

Then reordering the element of a_n you can obtain any number as the sum!

$$a_n = \frac{(-1)^n}{n}$$

$$a_n = (-1)^n$$

$A(s)$ The generating function exists for $|s| < 1$ and if $|s| < 1$ Then

$$A(s) = \frac{1}{1+s}$$

Probability Generating functions.

Suppose that X is a r.v.

such that

$$Im(X) \in \mathbb{N}$$

$$p_n = p_X(n) = \mathbb{P}(X=n)$$

After this I can define

$$G_X(s) = p_0 + p_1 s + p_2 s^2 + \dots = \sum_{n=0}^{\infty} p_n s^n$$

G_X is the prob. gen. funct of X

Theorem: if X and Y are two integer-valued function then

$$G_X(s) = G_Y(s) \quad \text{for all } s$$

if and only if

$$\mathbb{P}(X = n) = \mathbb{P}(Y = n) \quad \forall n$$

How can I write $G_X(s)$ has a expectation

$$G_X(s) = \sum_{n=0}^{\infty} \mathbb{P}(X = n)\, s^n =$$

$$= \mathbb{E}(s^X)$$

Suppose that X and Y are indep. Consider $X + Y$

$$G_{X+Y}(s) = \mathbb{E}(s^{X+Y}) =$$

$$= \mathbb{E}(s^X s^Y) =$$

$$= \mathbb{E}(s^X)\,\mathbb{E}(s^Y)$$

$$= G_X(s)\, G_Y(s)$$

$X : \Omega \to \mathbb{N}$

$s^X : \Omega \to \mathbb{N}$

$(s^X)(\omega) = s^{X(\omega)}$

Outcome of flipping 10 Times a coin

X is # of H.

$\underline{\omega}$ = seq. of 10 H T

$X(\underline{\omega})$ = # of H

$s^X : \Omega \to \mathbb{R}$

$(s^X)(\omega) = s^{X(\omega)}$

$$\mathbb{E}(s^X) = \sum_{n=0}^{\infty} \mathbb{P}(X=n)\, s^n$$

$$\mathbb{E}(f(X)) = \sum_{n=0}^{\infty} \mathbb{P}(X=n)\, f(n)$$

If X and Y are indep. integer valued r.v. Then

$$G_{X+Y}(s) = G_X(s)\, G_Y(s)$$

$$\mathbb{P}(X+Y=n) = \sum_m \mathbb{P}(X=m)\,\mathbb{P}(Y=n-m)$$

Suppose X and Y are Poisson.

μ ν

$$G_X(s) = \mathbb{E}(s^X) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} e^{-\mu} s^n =$$

$$= e^{-\mu} \sum_{n=0}^{\infty} \frac{(\mu s)^n}{n!} =$$

$$= e^{-\mu} e^{\mu s} =$$

$$= e^{\mu(s-1)}$$

$$G_Y(s) = e^{\nu(s-1)}$$

$$G_{X+Y}(s) = e^{\nu(s-1)} e^{\mu(s-1)} =$$
$$= e^{(\nu+\mu)(s-1)}$$

I see that $e^{(\nu+\mu)(s-1)}$ is the prob gen. funct of a Poissonian with par $\nu+\mu$.

By uniqueness follows that $X+Y$ is Poissonian par $\mu+\nu$!

Bernoulli

1	p
0	$(1-p)$

$$G_X(s) = s^0(1-p) + sp =$$
$$= (1-p) + sp = 1 + p(s-1)$$

Binomial

$$p(n) = \binom{N}{n} p^n (1-p)^{N-n}$$

$$G_X(s) = \sum_{n=0}^{N} s^n \binom{N}{n} p^n (1-p)^{N-n} =$$

$$= \sum_{n=0}^{N} \binom{N}{n} (sp)^n (1-p)^{N-n} =$$

$$= (sp + (1-p))^N$$

So X is Bernoulli par p

Y is Binomial par N, p

$$G_Y(s) = \left(G_X(s) \right)^N$$

X_i are i.i.d. random variables

N is a r.v. indep from The X_i

$$S = \sum_{i=1}^{N} X_i$$

Flip a coin: N position of

The first sequence of 3 H.

X_i is 1 if H

-1 if T

$$S(\omega) = \sum_{i=1}^{N(\omega)} X_i(\omega)$$

$$G_S(t) = \mathbb{E}(t^S) =$$

$$\sum_n \mathbb{P}(N=n)\, \mathbb{E}(t^S \mid N=n)$$

$$\sum_{n=0}^{\infty} \mathbb{P}(N=n) \left(G_{X_1}(t)\right)^n =$$

$$\sum_{n=0}^{\infty} \mathbb{P}(N=n) (s)^n \quad \text{where}$$

$$s = G_{X_1}(t)$$

$$= G_N(G_{X_1}(t))$$